Determining Elevation Angles

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I. Introduction

This problem involves determining the angle above the horizon for a distant object. An example is observing distant mountains above one's horizon. The mountains may in fact be beyond the observer's horizon but are visible because they rise well above sea level. The premise then for this exercise is to consider the object height (such as a mountain top) above sea level to be fixed. There are then two parameters that the observer can adjust that will determine whether the object will be in view or not. Namely, the observer's height above sea level and the distance between the object and observer. We will not only determine whether the object can be viewed or not by the observer, but also calculate the elevation angle: the angle of the object, in degrees, above or below the observer's horizon.

II. Elevation Angle Geometry

In order to determine the elevation angle of an object at an for observer a different position and height we must consider the necessary geometry th describes the situation. Here will only be consider whether an object is viewable; that is, potentially visible. An object that is *not viewable* will be defined as one that is below the observer's viewable horizon.

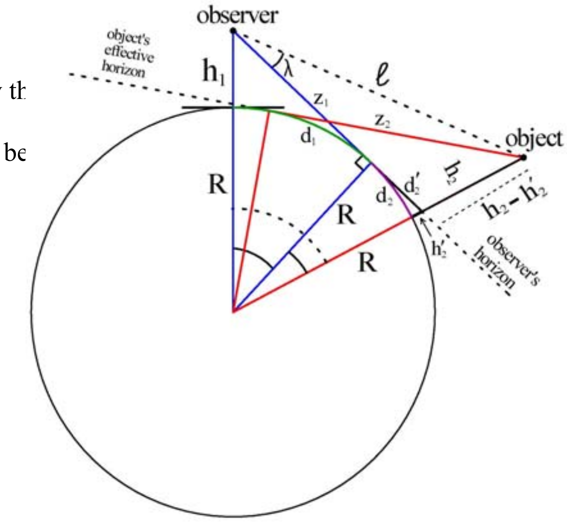


Fig. 1. Elevation angle (λ) geometry

Fig. 1 shows the geometry involved in determine the elevation angle. Parameters shown in Fig. 1 defined:

h_1: *observer* height above sea level.
h_2: *object* height above sea level.
d_1: distance to *observer's* horizon.
d_2 : distance form observer's horizon to the object
d : total distance from observer to object.
z_1: linear distance of *observer's* horizon (as viewed form observer's position).
z_2: linear distance of *object's* horizon (as viewed form object' position).
d_2': linear distance of object beyond observer's horizon.
h_2': Intersection of observers line of sight with height of object.
ℓ : line of sight from the position of observer to object's position.
R: radius of Earth = 3960 mi = 6378 km.
λ : elevation angle (angle of object above observer's horizon).

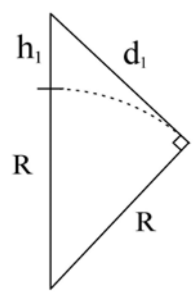


Fig. 2. Geometry from observer to horizon.

Based on Fig. 2, we can approximate to excellent accuracy that the opposite side is equal to the curved distance d_1. Thus from the right triangle shown in Fig. 2:

$$R^2 + d_1^{\,2} = \left(R + h_1\right)^2 \qquad (1)$$

$$R^2 + d_1^{\,2} = R^2 + 2Rh_1 + h_1^{\,2} \qquad (2)$$

We can assume that $h_1 << R$, therefore we can remove the h_1^2 term and also cancel th R^2 terms. Thus, the observer's horizon distance z_1 can be written as:

$$z_1 = \sqrt{2Rh_1} \qquad (3)$$

We can make the identical argument for determining the object's effective horizon distance z_2 (the horizon distance that would be observed from the position of the object):

$$z_2 = \sqrt{2Rh_2} \qquad (4)$$

Since the elevation angle λ is a small angle, we see, by excellent approximation, $d_2' << R$, thus $d_2' \sim d_2$. And since $h_1 << R$, $d_1 \sim z_1$, thus the *total distance* from the observer's position can be written as:

$$\boxed{d = z_1 + d_2} \qquad (5)$$

Where, d_2 is the from the observer's horizon to the object, as shown in Fig. 1. Our goal now is to determine the elevation angle λ as a function of h_1 , h_2 , and d. That is, what is the angle above the horizon of an object, such as a mountain top, for an observer at a particular elevation and distance from the object?

From Fig. 1, we can see the triangle represented in Fig. 3, where:

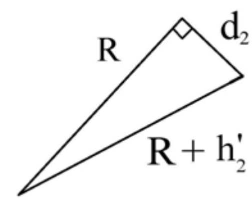


Fig. 3. Geometry over observer's horizon.

Expanding:

$$R^2 + d_2^{\,2} = \left(R + h_2'\right)^2 \qquad (6)$$

$$R^2 + d_2^{\,2} = R^2 + 2Rh_2' + h_2'^{\,2} \qquad (7)$$

Removing higher order term $(h_2')^2$ on right-hand side (since $h_2' << R$) and canceling the R^2 terms yields

$$d_2^{\,2} = 2Rh_2' \qquad (8)$$

Thus:

$$h_2' = \frac{d_2^{\,2}}{2R} \qquad (9)$$

From Eq. (5) we see that $d_2 = d - z_1$ so we can rewrite Eq. 9 as:

$$h_2' = \frac{(d - z_1)^2}{2R} \qquad (10)$$

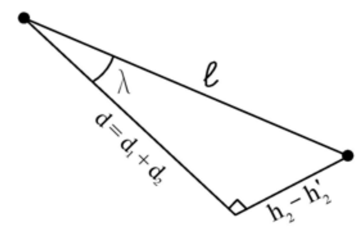


Fig. 4 Elevation angle λ.

III. Solution

From the geometry shown in Fig. 4 we consider λ ~ small.
Then to good approximation we can assume a right triangle, thus:

$$\tan(\lambda) = \frac{(h_2 - h_2')}{d} \qquad (11)$$

Since, typically, $h_2 - h_2' << d$ (height of the object is much less than the distance to object), $\tan(\lambda) \approx \lambda$.
Therefore, we can write to a good approximation:

$$\lambda = \frac{(h_2 - h_2')}{d} = \frac{h_2}{d} - \frac{h_2'}{d} \qquad (12)$$

By substitution from Eq. (10):

$$\lambda = \frac{h_2}{d} - \frac{\left(d - z_1\right)^2}{2Rd} = \frac{1}{2R}\left[\frac{2Rh_2}{d} - \frac{\left(d - z_1\right)^2}{d}\right] \qquad (13)$$

Where, from Eq. (4), $z_2^{\,2} = 2Rh_2$, thus:

$$\boxed{\lambda(rad) = \frac{1}{2R}\left[\frac{z_2^{\,2} - \left(d - z_1\right)^2}{d}\right]} \quad \text{*Elevation Angle Formula} \qquad (14)$$

Where:

z_1: distance to observer's horizon, $z_1 = \sqrt{2Rh_1}$.

z_2: distance of object's horizon (as viewed form object' position), $z_2 = \sqrt{2Rh_2}$.

d : total distance from the observer to the object in miles.

R: radius of Earth = 3960 mi

***Note that this solution yields λ in radians. To convert to degrees multiply result by 180º/π ~ 57.3º.**

IV. Limits

Now we can ask, what is the minimum distance $d_{\min}$ that an observer at height h_1 can be from an object of height h_2 and still remain viewable? This will occur when $\lambda = 0$ (object is just on the horizon). This means that if d (the distance from the observer to the object) is less than d_{min} the object will be below the horizon and not veiwable. Thus from the elevation angle formula, Eq. (14), our condition for $\lambda = 0$ is:

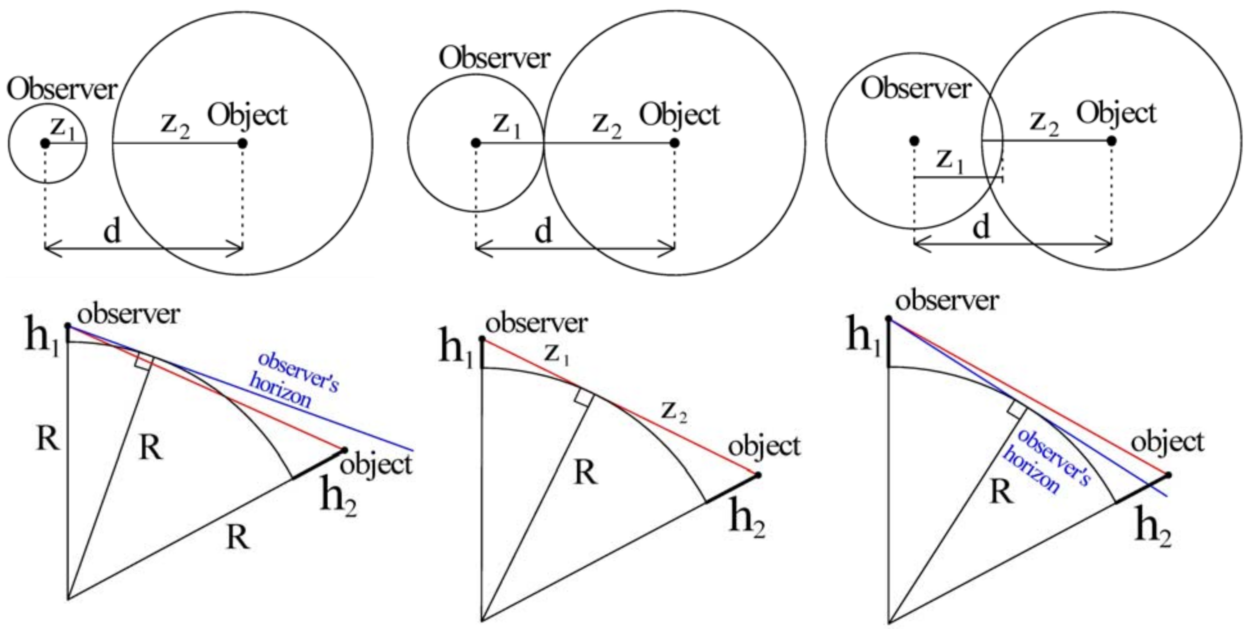
$$z_2^{\,2} - \left(d_{\min} - z_1\right)^2 = 0 \qquad (15)$$

Since $d \geq z_1$, {that is, our solution [Eq. (14)] assumes the object is on or beyond the observer's horizon} we find that: Our solution, based on Eq. 15, is that $d_{\min} - z_1 = z_2$, or

$$\boxed{d_{\min} = z_1 + z_2} \qquad (16)$$

This indicates that whether an object is viewable or not simply depends on the horizon distances of both the observer *and* the object, where z_1 and z_2 can be calculated by Eqs. (3) and (4).

For instance, if the observer is at sea level h_1=0, thus z_1=0 and therefor $d_{min} = z_2$. This indicates that if you are at sea level you must be within the *object's* horizon distance. If one is at sea level (z_1=0) and $d > d_{\min}$ the only way to possibly view the object would be to increase you elevation (h_1) until $z_1 + z_2 > d$. This is demonstrated below in the following three sets of diagrams shown in Fig. 5, below.



(a) $z_1 + z_2 < d$ (b) $z_1 + z_2 = d$ (c) $z_1 + z_2 > d$

Fig. 5. **Top figures:** Horizons of observer and object for an object a distance *d* away from observer. **Bottom figures:** Line of sight for various horizon configurations.

Fig. 5 shows how the line of sight varies for the corresponding horizon configurations. In sequence, the observer's height is increased, thus increasing the observer's horizon. In (a) the observer is at a relatively low height with a minimal horizon distance, thus the line of sight is below the observer's horizon ($\lambda < 0$). In (b) the observer's height is increased such that the observer's horizon is parallel with the object's horizon. In this case, it can be seen that the line of sight to the object is on the observer's horizon ($\lambda = 0$). In (c) the increased height causes the observer's horizon to overlap the object's horizon. In this case the object rises above the observer's horizon and is thus viewable ($\lambda > 0$).

V. Summary

It can be seen from Fig. 5 that whether an object is above the horizon and thus viewable simply depends on the sum of the observer's horizon z_1 and object's horizon z_2. In summary, the three conditions are as follows:

1. If $z_1 + z_2 < d$ the object is below the horizon and not viewable.
2. If $z_1 + z_2 = d$ the object is on horizon and not likely viewable.
3. If $z_1 + z_2 > d$ the object is above the horizon and viewable. The angle above the horizon can then be calculated by Eq. (14), the elevation angle formula. Whether object is actually viewable will also depend on seeing conditions and actual distance to the horizon. Objects over 100 miles away at sea level will become more and more difficult to see.